$$\chi(x) = \sum_{m=0}^{\infty} \frac{2F_m L_m^{\mu-1/2}(2x)}{e^x x^{1/2-\mu} \mu_m}, \qquad F_n = -\sum_{r=0}^{n} g_r I_{m-r} \tag{3.15}$$

For the integral I_m we have the formula (2.12) with the replacement of $G_+(p)$ by $K_+(p)$. In this case, for the latter one we can, apart of (3.7), make use of the expansion

$$K_+(p) = \sum_{m=0}^{\infty} g_m^* \left(\frac{p-i}{p+i}\right)^m \tag{3.16}$$

which follows from (3.11). Its use leads to the formulas (2.13), (2.14) and (2.16) with the replacement of g_n by g_n^*.

REFERENCES

1. Koiter, W. G., Approximate solution of Wiener-Hopf type integral equations with applications. Proc. Koninkl. nederl. akad. wet., Bd. 57, № 5, 1954.
2. Gokhberg, I. Ts. and Fel'dman, I. A., Convolution Equations and Projection Methods for Solving them. Moscow, "Nauka", 1971.
3. Popov, G. Ia., On integral equations of the theory of elasticity with difference and summation kernels. PMM Vol. 34, № 4, 1970.
4. Paskalenko, A. A. and Popov, G. Ia., Plane problem of bending of a semi-infinite beam resting on a linearly-deformable foundation. PMM Vol. 36, № 1, 1972.
5. Krein, M. G., Integral equations on the half-line with a kernel depending on the difference of the arguments. Uspekhi Matem. Nauk, Vol. 13, № 5, 1958.
6. Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity. Moscow, "Nauka", 1966.
7. Lanczos, C., Applied Analysis, Prentice Hall, Englewood Cliffs, 1956.
8. Popov, G. Ia., Bending of a semi-infinite plate resting on a linearly deformable foundation. PMM Vol. 25, № 2, 1961.

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OPERATOR METHOD OF INVESTIGATING THE STRAIN OF A HOLLOW SPHERE WITH DIFFERENT CREEP IN TENSION AND COMPRESSION

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The polar symmetric strain of a viscoelastic thick-walled hollow sphere, whose material possesses the property of different resistivity under tension and compression is considered. The vessel is subjected to internal pressure p and external tension p which are distributed uniformly over the surfaces $r = a$ and $r = b$ $(a < b)$. Because of the above, the vessel is separated into two parts by a spherical surface of radius $r = \rho$, which is independent of the quantity p during solution of the corresponding elastic problem [1] even when p varies with time t.

In solving the viscoelastic problem, let us proceed from the physical operator equations in spherical coordinates:

For the first part of the vessel $(\sigma_r > 0,\ \sigma_\theta = \sigma_\varphi > 0)$

$$E_t^+\varepsilon_r = \sigma_r - 2\nu_t^+\sigma_\theta, \qquad E_t^+\varepsilon_\theta = (1 - \nu_t^+)\sigma_\theta - \nu_t^+\sigma_r \tag{1}$$

For the second part $(\sigma_r < 0,\ \sigma_\theta = \sigma_\varphi > 0)$

$$E_t^-\varepsilon_r = \sigma_r - 2\nu_t^-\sigma_\theta, \qquad E_t^+\varepsilon_\theta = (1 - \nu_t^+)\sigma_\theta - \nu_t^+\sigma_r \tag{2}$$

Here σ_r, $\sigma_\theta = \sigma_\varphi$, $\varepsilon_\theta = \varepsilon_\varphi = u / r$, $\varepsilon_r = \partial u / \partial r$ are the stresses and strains, E_t^+, ν_t^+ and E_t^-, ν_t^- are the operator moduli and operator Poisson's ratios under tension (plus superscript) and compression (minus superscript), $u = u_1$ and $u = u_2$ are normal displacements in the first and second parts of the sphere, respectively, where the operators are

$$E_t^+ = E_0^+(1 - R_1^*), \qquad E_t^- = E_0^-(1 - R_2^*) \tag{3}$$

$$\nu_t^+ = \nu_0^+(1 - g_0^+R_1^*), \qquad \nu_t^- = \nu_0^-(1 + g_0^-R_2^*) \tag{4}$$

$$g_0^\pm = \frac{1 - 2\nu_0^\pm}{2\nu_0^\pm}, \quad R_i^*\{\cdot\} = \int_0^t R_i(t, \tau)\{\cdot\}\, d\tau \quad (i = 1, 2)$$

and R_i^* are integral operators with relaxation kernels $R_1(t, \tau)$ and $R_2(t, \tau)$ for tension and compression, respectively, E_0^+, ν_0^+ and E_0^-, ν_0^- are elastic instantaneous constants characterizing the different moduli of the material.

As a corollary of the operator expressions (3), we have

$$\frac{1}{E_t^+} = \frac{1}{E_0^+}(1 + P_1^*), \qquad \frac{1}{E_t^-} = \frac{1}{E_0^-}(1 + P_2^*)$$

where P_i^* are integral operators with the creep kernels $P_i(t, \tau)$. Hence $P_i^*R_i^* = P_i^* - R_i^*$. The expressions (4) correspond to the case of constant dilatation operator [2] under tension and compression.

In conformity with the propositions in [1, 3], it is assumed that

$$\nu_t^+ / E_t^+ = \nu_t^- / E_t^- \tag{5}$$

The generality of the operator method is in no way associated with condition (5). Just as the condition of no after-effect during dilatation condition (5) only contributes to obtaining less awkward results than in the general case when the number of relaxation kernels is greater than two.

It follows from physical considerations that $R_i^*\cdot 1$ and $P_i^*\cdot 1$ are positive monotonely increasing functions of time t, where $0 \leqslant R_i^*\cdot 1 \leqslant 1$ and $0 \leqslant P_i^*\cdot 1 < \infty$ as $t \to \infty$. The boundary conditions and conditions on the surface $r = \rho$ are the following:

$$\sigma_r(a, t) = -p,\ \sigma_r(b, t) = p,\ \sigma_r(\rho, t) = 0,\ u_1(\rho, t) = u_2(\rho, t) \tag{6}$$

In general, ρ depends on the time and is to be determined.

Solving (2) for σ_r and σ_θ and taking account of (5), we obtain

$$\Delta_t\sigma_r = 2\nu_t^+\frac{u_2}{r} + (1 - \nu_t^+)\frac{\partial u_2}{\partial r}, \qquad \Delta_t\sigma_\theta = \delta_t\frac{u_2}{r} + \nu_t^+\frac{\partial u_2}{\partial r} \tag{7}$$

$$\Delta_t = (1 - \nu_t^+ - 2\nu_t^+\nu_t^-) / E_t^-, \qquad \delta_t = E_t^+ / E_t^- \tag{8}$$

Equations corresponding to the first part of the sphere are obtained from (7) and (8) if we take $\nu_t^+ = \nu_t^-$, $E_t^+ = E_t^-$ therein and replace u_2 by u_1.

The following statements are proved below.

Theorem 1. If neither of the operators ν_t^+ and ν_t^- degenerates into a constant and the pressure p depends on time, then the radius of curvature $\rho(t)$ depends on the pressure $p(t)$ and is determined from a nonlinear integral equation.

Theorem 2. If $E_t^+ \nu_0^- = E_t^- \nu_0^+$, then the radius of curvature ρ is found from an algebraic equation corresponding to the elastic problem, is independent of time and pressure $p(t) = p_0 f(t)$, where p_0 is the initial pressure and $f(t)$ is a specified positive bounded monotonic function of time.

The displacements $u_1(t)$ and $u_2(t)$ vary in proportion to the elastic displacements $u_1(0)$ and $u_2(0)$ with the proportionality coefficient $D(t) = (1 + P^*_1) f(t)$ and as $t \to \infty$ approaches the limit values

$$u_i(\infty) = u_i(0)\, D(\infty) \quad \left(D(\infty) = f(\infty) + \int_0^\infty P_1(t, \tau) f(\tau)\, d\tau < \infty\right)$$

To prove these theorems it is sufficient to find the principal solution of the viscoelastic problem under consideration. Substituting the expressions for σ_r and σ_θ from (7), taking account of (8) and (5), into the equilibrium equation $r(\partial\sigma_r / \partial r) + 2(\sigma_r - \sigma_\theta) = 0$, we obtain the integro-differential equation

$$r^2 \frac{\partial^2 u_2}{\partial r^2} + 2r \frac{\partial u_2}{\partial r} - 2\mu_t u_2(r, t) = 0 \tag{9}$$

$$2\mu_t = \frac{2E_t^+ (1 - \nu_t^+)}{E_t^- (1 - \nu_t^+)} = 2\mu_0 + K^*, \qquad \mu_0 = \frac{E_0^+ (1 - \nu_0^-)}{E_0^- (1 - \nu_0^+)}$$

$$K^* = 2\mu_0 (q_0^- P_2^* - H_1^* - q_0^- H_1^* P_2^*), \qquad q_0^\pm = 1 / 2(1 - \nu_0^\pm)$$

The kernel $H_1(t, \tau)$ of the operator H^*_1 is the resolvent of the kernel $q_0^\pm P_1(t, \tau)$ of the operator $q_0^+ P^*_1$. Substituting

$$v(r, t) = r^2 \frac{\partial^2 u_2}{\partial r^2} + 2r \frac{\partial u_2}{\partial r} - 2\mu_0 u_2$$

reduces (9) to the integral equation

$$v = f_1 + \frac{1}{2n} K^* Q^{**} v \left(Q^{**} v = \int_a^r Q(r, \xi)\, v(\xi, \tau)\, d\xi\right) \tag{10}$$

The kernel of the coordinate operator Q^{**} and the function f_1 have the form

$$Q(r, \xi) = \frac{1}{2n}\left[\left(\frac{r}{\xi}\right)^{-\alpha_1} - \left(\frac{r}{\xi}\right)^{-\alpha_2}\right]\xi, \qquad f_1 = K^*(C_1 r^{-\alpha_1} + C_2 r^{-\alpha_2})$$

$$\alpha_i = 1/2 + (-1)^i n, \quad i = 1, 2, \quad n^2 = 1/4 + 2\mu_0$$

($C_1(t)$ and $C_2(t)$ are functions of time to be determined). Solving (10), we obtain $v = (1 + M) f_1$, where the operator M is expressed by the Neumann series

$$M = \sum_{m=1}^{\infty} \left(-\frac{1}{2n}\right)^m K^{*m} Q^{**m}$$

which converges uniformly, as is known [4], when $a \leqslant r \leqslant b$, $0 \leqslant t < \infty$. Here the kernels of the operators K^{*m} and Q^{**m} have the sense of repeated kernels

$$K_{m+1}(t, \tau) = \int_\tau^t K_1(t, s)\, K_m(s, \tau)\, ds$$

$$Q_{m+1}(r,\xi)=\int_{\xi}^{r} Q_1(r,\eta)\,Q_m(\eta,r)\,d\eta$$

$$K(t,\tau)=K_1(t,\tau),\quad Q(r,\xi)=Q_1(r,\xi)$$

Returning to the desired function $u_2(r,t)$, we obtain

$$u_2(r,t)=\psi_1(r)\ C_1(t)+\psi_2(r)\,C_2(t) \tag{11}$$

$$\psi_i(r)=r^{-\alpha_i}+\sum_{m=0}^{\infty}\varphi_{m+1}^{(i)}(r)\,K^{*m+1}$$

$$\varphi_{m+1}^{(i)}(r)=\frac{(-1)^m}{(2n)^{m+1}}\int_a^r Q_{m+1}(r,\xi)\,\xi^{-\alpha_i}\,d\xi$$

Using the conditions $\sigma_r(a,t)=-p$, $\sigma_r(\rho,t)=0$ and the first equation in (7) as well as (11), we obtain a system of linear integral equations

$$\lambda_1(a)\,C_1+\lambda_2(a)\,C_2+\Delta_t p=0,\qquad \lambda_1(\rho)\,c_1+\lambda_2(\rho)\,c_2=0 \tag{12}$$

Here

$$\lambda_i(a)=a^{-\alpha_i-1}(\delta_{1t}-\alpha_i\delta_{2t}),\quad \delta_{1t}=2\nu_t^+,\quad \delta_{2t}=1-\nu_t^+$$

$$\lambda_i(\rho)=(\delta_{1t}-\alpha_i\delta_{2t})\,\rho^{-\alpha_i-1}+\eta_i^*(\rho)$$

$$\eta_i^*(\rho)=\sum_{m=0}^{\infty}K^{*m+1}\left(\delta_{1t}\frac{1}{\rho}+\delta_{2t}\frac{\partial}{\partial\rho}\right)\varphi_{m+1}^{(i)}(\rho)$$

From the system (12) it follows that

$$L_tC_1+\Lambda(\rho)\,\Delta_t\,p=0,\qquad L_tC_2=\Delta_t p \tag{13}$$

$$L_t=\lambda_1(a)\,\Lambda_t(\rho)-\lambda_2(a),\qquad \Lambda_t(\rho)=\lambda_2(\rho)\,/\,\lambda_1(\rho)$$

The general solution of the differential equation determining the displacement $u_1(r,t)$ is

$$u_1=Ar^{-2}+Br \tag{14}$$

Replacing the quantities C_1, C_2 and a in the system (12) by A, B and b, respectively, and assuming

$$K^*\equiv 0,\ \alpha_1=2,\ \alpha_2=-1,\ \Delta_t=(1-\nu_t^+-2\nu_t^{+2})\,/\,E_t^+$$

we obtain a system of integral equations determining A and B. As a corollary of its solution and the representation (14), we have

$$u_1(r,t)=1/2b^3(\rho^3r^{-2}+2rT_{2t})(b^3-\rho^3)^{-1}T_{1t}p \tag{15}$$

$$T_{1t}=\frac{1+\nu_t^+}{E_t^+},\qquad T_{2t}=\frac{3}{1+\nu_t^+}-2$$

Using the condition $u_1(\rho,t)=u_2(\rho,t)$, and taking account of the representations (11), (13) and (15), we obtain the nonlinear integral equation

$$b^3L_t\rho(1+2T_{2t})\ (b^3-\rho^3)^{-1}T_{1t}p=2\ [\psi_1(\rho)\,\Lambda_t(\rho)-\psi_2(\rho)]\ \Delta_t p \tag{16}$$

All the effects mentioned here are satisfied in conformity with the theory developed in [5] and by means of the rules established in [6, 7]. Equation (16) can be solved by successive approximations. The structure of (16), determining the quantity $\rho(t)$, verifies the validity of Theorem 1 and shows that the Volterra principle cannot be used to solve the problem under consideration in the general case.

When $\nu_t^+ = \nu_0^+$ and $\nu_t^- = \nu_0^-$, then by virtue of the property (5) the ratio of the operators E_t^+ / E_t^- degenerates into the constant ν_0^+ / ν_0^-. Since $\mu_t = \mu_0$ here, then the integro-differential equation (9) goes over into a known differential equation and the system of integral equations degenerates into an algebraic system. The nonlinear integral equation (16) hence reduces to the known algebraic equation [1] which does not contain the pressure p. Only the displacements $u_1(t)$ and $u_2(t)$ which are obtained by replacing the quantity $1/E_t^+$ in (7.13) and (7.15) in [1] by

$$1/E_t^+ = (1/E_0^+)(1 + P^*_1)$$

will depend on time.

This latter confirms the validity of Theorem 2 and shows that the Volterra principle is valid in this particular case. The condition $\mu_t = \mu_0$ is not itself sufficient for applicability of the Volterra principle since it can also be realized for $\nu_t^+ \neq \nu_0^+$ and $\nu_t^- \neq \nu_0^-$. In this case only simplification of the problem occurs.

REFERENCES

1. Ambartsumian, S. A. and Khachatrian, A. A., Fundamental equations of the theory of elasticity for materials differently resistive to tension and compression. Inzh. Zh., Mekh. Tverd. Tela, №2, 1966.
2. Rabotnov, Iu. N., Equilibrium of an elastic medium with after-effect. PMM Vol.12, №1, 1948.
3. Ambartsumian, S. A., On a model of a hereditarily elastic solid differently resistive to tension and compression. PMM Vol. 35, №1, 1971.
4. Miunts, G., Integral Equations. ONTI, Moscow-Leningrad, 1934.
5. Krasnosel'skii, M. A., Positive Solutions of Operator Equations. Fizmatgiz, Moscow, 1962.
6. Rozovskii, M. I., On a class of functions and their applications, (English translation). Pergamon Press, J. USSR Comput. Mat. mat Phys., Vol. 2, №1, 1962.
7. Datkaev, A. M. and Rozovskii, M. I., On operator and operator-differential equations of hereditary creep theory. Izv. Akad. Nauk ArmSSR, Mekhanika, Vol.19, №1, 1966.

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